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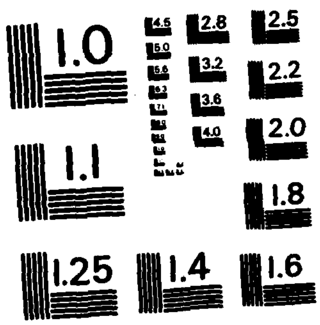

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ITEM #20, CONTINUED: state variable formulation. The author also shows that this dependent demand model orders less than or equal the amount that a comparable independent demand model orders. This result is established under the assumption that all demand is returned by the beginning of the next period.

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SCARF'S STATE REDUCTION METHOD, FLEXIBILITY,

AND A DEPENDENT DEMAND INVENTORY MODEL

by

Bruce L. Miller

April, 1983

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ABSTRACT

We consider a finite horizon inventory model where the holding, shortage, and ordering costs are linear. The demand random variables are dependent and average demand is described by an exponential smoothing formula. This model can be formulated as a two state variable (inventory level, weighted past demands) dynamic program. By using a procedure first developed by Scarf for a Bayesian inventory model, we are able to reformulate the model as a one state variable dynamic program. This, of course, results in a considerable computational saving over the two state variable formulation. We also show that this dependent demand model orders less than or equal the amount that a comparable independent demand model orders. This result is established under the assumption that all demand is returned by the beginning of the next period.

I. INTRODUCTION

The study of inventory theory is often divided into three parts: deterministic models, stochastic models, and forecasting (Sections 12.3, 12.4, and 12.5 respectively of Hillier and Lieberman [1980]). The stochastic models generally assume that the demand distributions of inventory in the different periods are independent. However, this independence assumption is at odds with the demand forecasting methods which are consistent only with a dependent hypothesis. This suggests that the assumption that inventory demands in different periods are independent is often unrealistic, and that more attention should be given to the dependent demand case. The computational approach to the dependent demand problem is evident from dynamic programming concepts, and consists of adding a second state variable pertaining to demand (the first being the inventory level) in the equation of optimality. This greatly increases the computational burden and helps explain why the dependent demand case had not received more attention.

The assumption of dependent inventory demands has been made in the recent models of Blinder [1982], Pindyck [1982], and Harpaz, Lee, and Winkler [1982]. Each paper observed that earlier related papers made the assumption of demand independence.

Blinder [1982,p.337] writes:

> "Each of the papers referred to above assumes that demand shocks are independently and identically distributed. This assumption, while it simplifies things greatly, is quite unsatisfactory. We know, for example, that disturbances at the macro level are highly serially correlated, and it would be surprising indeed if this serial correlation disappeared when we disaggregated to the industry or firm levels."

Blinder assumed an autoregressive demand process of order 1. Pindyck represented demand as a multiplicative function of Brownian motion, and Harpaz, Lee, and Winkler used a Bayesian demand model. Those papers were concerned with the economic properties of their models, while we emphasize the computational aspects as well as qualitative properties of the inventory problem.

This paper will assume that the expected value of demand is given by an exponential smoothing formula, and that uncertainty is multiplicative. Two earlier efforts at combining a dependent demand model and the computation of inventory ordering policies are Packer [1967] and Johnson and Thompson [1975]. Packer considers an exponential smoothing forecasting model. In that paper several easy to compute policies are compared by simulation. Johnson and Thompson [1975] consider a Box Jenkins forecasting model. They obtain conditions on the demand parameters and cost structure such that a myopic policy is optimal. Their approach uses the results of Veinott [1965], and consists of showing that beginning inventory at each period cannot be larger than the desired (myopic) level in that period. When it applies, this result is very powerful computationally and cannot be improved upon. A two state dynamic program has been reduced to a static model (a "zero state" dynamic program).

A main result of this paper (Theorem 1) shows how a two state variable dynamic program can be reduced to one state variable under our demand and cost structure. Thus a dependent demand case becomes no more difficult to solve than the independent demand case. To do this we use the method of Scarf [1960a], who was able to reduce a two state variable Bayesian inventory model to one state variable when demand is gamma. Azoury [1979] has extended Scarf's Bayesian inventory result to the cases where demand is uniform and Weibull.

In the last section we consider the concept of flexibility. Following Marshak and Nelson [1962] we say that an action a is more flexible than action

b if the next period's set of feasible actions is larger when action a is used. In Theorem 2 it is shown that our dependent demand model results in more flexible actions than a comparable independent demand model. This result is established for the case where all demand is returned by the beginning of the next period.

II. THE MODEL

As in Scarf [1960a] we assume a linear ordering cost, a linear shortage cost and a linear holding cost. These are standard assumptions the most restrictive being the linear ordering cost.

Our formulation of inventory behavior follows Cohen, Pierskalla, and Nahmias [1980]. We allow a fixed fraction of stock, $1-\beta$, held at the end of each review period to decay. We also allow a fixed fraction of demand, $1-\alpha$, to be returned by the beginning of the next period. This fraction could represent the proportion of demands that are repairable in a repairable inventory model where demand represents past failures (see the example Sherbrooke [1968] or Miller [1974]). In military applications the times between orders are typically longer than the repair times so the assumption that repaired items have returned by the next period is realistic. Cohen, Pierskalla, and Nahmias [1980] are more general and permits a fixed fraction of demand to be returned after a delay of $\gamma \geq 1$ periods. They allow no backlogging while this model permits any fraction δ of backlogging.

Let x_i represent the inventory level at the beginning of period i, y_i represent the inventory level in period i immediately after ordering $(y_i-x_i) \geq 0$ units, and z_i represent the quantity demanded during period i. According to the assumptions of the previous paragraph, for $i=1,\ldots,n$,

$$x_{i+1} = \begin{cases} \beta(y_i-\alpha z_i) & \text{if} \quad y_i - \alpha z_i \geq 0 \\ \delta(y_i-\alpha z_i) & \text{if} \quad y_i - \alpha z_i < 0 \end{cases} \tag{1}$$

Our formulation of inventory demand is influenced by Brown [1959]. Let Δ_i, $i=1,\ldots,n$, be the "average demand factor" in period i. The Δ_i evolve according to the following exponential smoothing formula

$$\Delta_1 = \mu, \text{ and}$$
$$\Delta_i = (1-e_{i-1})\,\Delta_{i-1} + e_{i-1}z_{i-1} \qquad 2 \leq i \leq n. \tag{2}$$

In (2) μ is an a priori estimate of average demand in period 1, e_i, $0 \leq e_i \leq 1$, is a smoothing constant, and z_i, as previously defined, is the quantity demanded in period i. Let Z_i represent the demand random variable in period i. On p. 94 of [1959] Brown states, "You will be very likely to find that the standard deviation of demand is nearly proportional to the total annual usage, or to the average monthly usage." A formulation of demand consistent with this observation is

$$Z_i = \Delta_i A_i \qquad 1 \leq i \leq n \tag{3}$$

where the A_i are independent nonnegative random variables. Often $E[A_i] = 1$ so that $E[Z_i] = \Delta_i$, but if demand is expected to be increasing or decreasing then $E[A_i] > 1$ or $E[A_i] < 1$. We let F_i be the distribution function of A_i and $G_i(\cdot|\Delta_i)$ be the distribution function of Z_i.

Clearly $\Delta_i \geq 0$ and hence $Z_i \geq 0$. When uncertainty is additive instead of multiplicative restrictions must be placed on the parameters to ensure that demand cannot become negative (see Johnson and Thompson [1975,p.1306]). If the e_i are all zero then we have the standard model where the Z_i are independent, while if the e_i are all one then by induction $Z_j = \mu A_1 \ldots A_j$ and $\ell n Z_j$ is a random walk when the A_i are identically distributed. In this case we have a discrete time version of the lognormal demand hypothesis in Pindyck [1982,footnote 3]. Brown devotes Appendix C in [1959] to the lognormal distribution and its applicability to inventory demand.

Let c_i, h_i, and p_i represent the linear ordering, holding, and shortage costs in period i respectively. The value of p_i will usually depend on the assumptions made about δ. Allowing the costs to vary with the period permits us to incorporate any desired discounting or inflation factor into the model. The expected holding and shortage cost is given by the familiar formula

$$\begin{aligned} L_i(y|\Delta) &= h_i \int_0^y (y-s)\, dG_i(s|\Delta) + p_i \int_y^\infty (s-y)\, dG_i(s|\Delta) \qquad y \geq 0 \\ &= p_i \int_0^\infty (s-y)\, dG_i(s|\Delta) \qquad y < 0. \end{aligned} \tag{4}$$

Let $V^o(\cdot)$ be the piecewise linear salvage value of inventory after period n. We assume that

$$\begin{aligned} V^o(x) &= k_1 x \qquad x \geq 0 \\ &= k_2 x \qquad x < 0 \end{aligned} \tag{5}$$

where $$k_2 \geq k_1 > 0.$$

The equations (1-5) enable us to write the dynamic programming equation of optimality

$$V_i(x,\Delta) = \min_{y \geq x}\Big[c_i(y-x) + L_i(y|\Delta) + \int_0^{y/\alpha} V_{i+1}(\beta(y-\alpha s),\ (1-e_i)\Delta+e_i s)\ dG_i(s|\Delta) + \int_{y/\alpha}^\infty V_{i+1}(\delta(y-\alpha s),\ (1-e_i)\Delta+e_i s)\ dG_i(s|\Delta)\Big]$$

with $$V_{n+1}(x,\Delta) = V^o(x).$$

If $\alpha = 0$ then $y/\alpha = \infty$ and the second integral disappears. The interpretation of $V_i(x,\Delta)$ is the expected cost over periods $i,\ldots,n+1$, using an optimal policy with an inventory level x and an average demand factor Δ.

III. COMPUTING AN OPTIMAL POLICY

In this section we will show that our model can be solved as a one state

variable dynamic program. This will be accomplished by applying the method Scarf [1960a] used to reduce a two state variable Bayesian inventory model to one state variable. This, of course, represents a considerable computational saving.

The idea of the proof can be gathered from the following simple lemma.

<u>Lemma 1</u>. $L_i(y|\Delta) = \Delta \mathscr{L}_i(y|\Delta)$ where

$$\mathscr{L}_i(y) = h_i \int_0^y (y-t)\ dF_i(t) + p_i \int_y^\infty (t-y)\ dF_i(t) \qquad y \geq 0$$

$$= p_i \int_0^\infty (t-y)\ dF_i(t) \qquad y < 0.$$

<u>Proof</u>. We consider only the case $y \geq 0$ as the other case is easier. By (3) and (4),

$$L_i(y|\Delta) = h_i \int_0^{y/\Delta} (y-\Delta t)\ dF_i(t) + p_i \int_{y/\Delta}^\infty (\Delta t-y)\ dF_i(t)$$

$$= h_i\Delta \int_0^{y/\Delta} (y/\Delta-t)\ dF_i(t) + p_i\Delta \int_{y/\Delta}^\infty (t-y/\Delta)\ dF_i(t)$$

$$= \Delta \mathscr{L}_i(y/\Delta). \quad \text{Q.E.D.}$$

<u>Theorem 1</u>. For $i=1,\ldots,n$, $V_i(x,\Delta) = \Delta W_i(x/\Delta)$ where $W_i(\cdot)$ is defined by

$$W_i(x) = \min_{y \geq x} \left[c_i(y-x) + \mathscr{L}_i(y) + \int_0^{y/\alpha} (1-e_i+e_it)\ W_{i+1}\left(\frac{\beta(y-\alpha t)}{1-e_i+e_it}\right) dF_i(t) + \int_{y/\alpha}^\infty (1-e_i+e_it)\ W_{i+1}\left(\frac{\delta(y-\alpha t)}{1-e_i+e_it}\right) dF_i(t) \right]$$

with $$W_{n+1}(x) = V^o(x). \tag{7}$$

<u>Proof</u>. By (5) it - -lea- ..at Theorem 1 holds for $i=n+1$. We assume it holds for $i+1$ and shows that it holds for i. Using Lemma 1, the induction hypothesis, and (3), (6) can be written as

$$V_i(x,\Delta) = \min_{y\geq x}\left[c_i(y-x) + \Delta\,\mathscr{L}_i(y/\Delta) + \int_0^{y/\Delta\alpha} \Delta(1-e_i+e_it)\, W_{i+1}\left(\frac{\beta(y-\alpha\Delta t)}{\Delta(1-e_i+e_it)}\right)dF_i(t)\right.$$

$$\left. + \int_{y/\Delta\alpha}^{\infty} \Delta(1-e_i+e_it)\, W_{i+1}\left(\frac{\delta(y-\alpha\Delta t)}{\Delta(1-e_i+e_it)}\right) dF_i(t)\right].$$

In the previous equation we also use $\Delta_{i+1} = (1-e_i)\,\Delta_i + e_iz_i = (1-e_i)\,\Delta_i + e_i\Delta_ia_i$ where a_i is the realization of A_i.

By factoring out Δ,

$$V_i(x,\Delta) = \Delta \min_{y\geq x}\left[c_i(y/\Delta-x/\Delta) + \mathscr{L}_i(y/\Delta) + \int_0^{y/\Delta\alpha} (1-e_i+e_it)\, W_{i+1}\left(\frac{\beta(y/\Delta-\alpha t)}{1-e_i+e_it}\right)dF_i(t) + \int_{y/\Delta\alpha}^{\infty} (1-e_i+e_it)\, W_{i+1}\left(\frac{\delta(y/\Delta-\alpha t)}{1-e_i+e_it}\right)dF_i(t)\right].$$

Let $r = y/\Delta$ and $q = x/\Delta$. Then $V_i(x,\Delta) = \Delta \min_{r\geq q}\left[c_i(r-q) + \mathscr{L}_i(r) + \int_0^{r/\alpha} (1-e_i+e_it)\, W_{i+1}\left(\frac{\beta(r-\alpha t)}{1-e_i+e_it}\right)dF_i(t) + \int_{r/\alpha}^{\infty} (1-e_i+e_it)\, W_{i+1}\left(\frac{\delta(r-\alpha t)}{1-e_i+e_it}\right)dF_i(t)\right]$

$= \Delta\, W_i(q)$. Q.E.D.

It is well known that the solution to an inventory equation such as (7) is not difficult and consists of solving successively for the critical levels S_i, where S_i is the value of y which minimizes the term in brackets in (7) (with x fixed at some very low level). Once these levels have been computed the optimal ordering policy in period i is

order	$S_i\Delta_i - x_i$	if	$x_i < S_i\Delta_i$
order	nothing	if	$x_i \geq S_i\Delta_i$.

On page 592 of [1960a] Scarf states but does not give a proof that his state reduction procedure also applies to the case of a fixed lead time of λ periods from the time of order to receipt of the goods. We now outline the argument which shows that our model also can solve the lead time case with one state variable. In what follows β and δ are both one.

The arguments which result in a reduction from λ state variables to one state variable in the lead time case with independent demands can be found in several places including page 201 of Scarf [1960b]. The state variable is u_i, the stock level in period i plus the amounts already ordered but not yet delivered. The relevant demand is the sum of demands in periods i, i+1,...,i+λ.

For our model with dependent demands and a fixed lead time of λ periods, it appears that two state variables may be needed in the equation of optimality, u_i and Δ_i. However, we can carry out the same reduction as before to one state variable, u_i/Δ_i, if $Z_i + \sum_{j=1}^{\lambda} Z_{i+j}$ can be written as Δ_i times some random variable. This is the case, as can be shown by an elementary but somewhat tedious demonstration. For example for $\lambda = 1$, $Z_i + Z_{i+1} = \Delta_i A_i + \Delta_{i+1} A_{i+1} = \Delta_i A_i + \left[(1-e_i)\,\Delta_i + e_i \Delta_i A_i\right] A_{i+1} = \Delta_i\left(A_i + (1-e_i)\,A_{i+1} + e_i A_i A_{i+1}\right)$. This last calculation suggests that the lead time model with dependent depends, although reducible to one state variable, could be computationally formidable.

IV. THE FLEXIBILITY OF AN OPTIMAL POLICY

The optimal policy of our dependent demand model when compared to the optimal policy of the standard inventory model with independent demands has an interesting property. In order to anticipate and interpret this property let us consider sequential decision models in general and quote from Marshak and Nelson [1962,p.42].

> "Many decision problems are characterized by the following structure:

(a) The payoff to the decision maker is a function of a sequence of actions taken by him, at times $t_1, t_2, \ldots, t_n, \ldots$, and a sequence of states of the world beyond his control.

(b) At any point in the decision sequence, the decision maker has less then perfect information about what the future states of the world will be.

(c) Although before time t_n the decision maker is uncertain as to what the world will be like at time t_n, he is less uncertain at times closer to t_n than he was at times farther away. The decision maker acquires additional information — he learns about future states of the world — as time goes by."

When we recall Hillier and Lieberman's partition of inventory theory, mentioned in the opening sentence of this paper, we see that the deterministic inventory models satisfy Marshak and Nelson's (a) only, the stochastic inventory models satisfy (a) and (b) only, while the model of this paper with its demand process given by (2) and (3) has all three characteristics (a), (b), and (c).

Marshak and Nelson (1962) point out that while many decision problems have characteristics (a), (b), and (c), they are usually not easy to solve and heuristics are often used. Marshak and Nelson suggest that flexibility is a desirable property of a policy, and they give three definitions of flexible actions. We will use their first definition which says that an action a is more flexible than an action b if the set of feasible actions in the next period when using action a is larger than the set of feasible actions in the next period when using action b. In our inventory problem the feasible region in

(6) and (7) is $y \geq x$, so that a lower value of inventory ordered is a more flexible decision than a higher value of inventory ordered. The feasible region in the next period will be larger if less inventory is ordered this period. Marshak and Nelson (1962) show in 3 2-period examples how flexibility becomes more important the more sequential decision models have characteristic (b) and especially (c).

Henry [1974] considered the idea of flexibility or more precisely its opposite, an "irreversible decision." He showed (Proposition 1) that if a model is simplified by replacing all random variables by their means, then the simplified model will more readily choose an inflexible "irreversible decision" than the original model would. Operationally the point that Henry and Marshak and Nelson are making is that there is a danger that heuristics and simplifications in sequential decision models will bias us toward the more inflexible decision. Another notable paper on flexibility is by Kreps [1979] who shows in a general setting the equivalence of the desire for flexibility and future uncertainty. Merkhofer [1977] looks at the converse issue, how greater flexibility in decision making increases the desire for better information.

Our objective in this section will be to compare the optimal policy of the inventory model which we have considered and will now call the Dependent Demand Model with the optimal policy of a comparable Standard Model where demand is independent each period. The previous work on flexibility just cited suggests that the Dependent Demand Model which satisfies Marshak and Nelson's (c) will order less inventory (be more flexible) than the Standard Model which does not satisfy (c). We will prove this result for the case where $1-\alpha$, the fixed function of demand returned at the beginning of the next period is 1. We now turn to formulating the comparable Standard Model.

The Standard Model will, of course, have the same cost structure as the Dependent Demand Model. The independent random variables representing demand

will be called B_i, $i=1,\ldots,n$, and have the same distribution as the unconditional distribution of the Z_i. Thus the distribution function of B_i equals $E_{\Delta_i}(G_i(\cdot\,|\Delta_i)) \triangleq G_i(\cdot)$. The expected holding and shortage cost each period for the Standard Model is

$$h_i \int_0^y (y-s)\ dG_i(s) + p_i \int_y^\infty (s-y)\ dG_i(s) \qquad y \geq 0$$

which equals

$$E_{\Delta_i}\left(h_i \int_0^y (y-s)\ dG_i(s|\Delta_i) + p_i \int_y^\infty (s-y)\ dG(s|\Delta_i)\right)$$

$$= E_{\Delta_i}(L_i(y|\Delta_i)) \triangleq L_i(y). \tag{8}$$

This equation also holds for $y < 0$.

Let $U_i(x)$ be the expected cost over periods $i,\ldots,n+1$, for the Standard Model using an optimal policy with inventory level x. Recall that α is assumed to equal 0. The equation of optimality for U_i is

$$\begin{aligned} U_i(x) &= \min_{y\geq x}\left[c_i(y-x) + L_i(y) + \int_0^\infty U_{i+1}(\beta y)\ dG_i(s)\right] \\ &= \min_{y\geq x}\left[c_i(y-x) + L_i(y) + U_{i+1}(\beta y)\right] \end{aligned} \tag{9}$$

with $\qquad U_{n+1}(x) = V^o(x)$.

Since the cost structure and the optimal return functions U_i and W_i are convex, (see, for example, Scarf (1960b)) the following results on convexity in Rockafeller [1970, Theorems 23.1, 24.1, 24.2, Corollary 24.2.1] will be useful.

Let $f = X \to R$ be convex where X is a convex subset of R.

Then

(a) the right-hand and left-hand derivatives of f, f'_+ and f'_-, exist everywhere on the interior of X, are nondecreasing, and satisfy $f'_+ \geq f'_-$.

(b) for any a, b, ε X

$$f(b) - f(a) = \int_a^b f'_+(s)ds = \int_a^b f'_-(s)ds.$$

Let us assume that the minimum of $f: X \to R$ exists. It is clear from (a) and (b) that f achieves a minimum at the point

$$x^* = \sup\{x : f'_+ \leq 0\} \tag{10}$$

In what follows it is understood that if the derivative of a convex function does not exist then we mean the right hand derivative. Thus in effect our derivatives are really right hand derivatives which exist everywhere on the interior of X. This use of right hand derivatives causes no problems since our optimality condition (10) is in terms of right hand derivatives.

Let

$$J_i^u(y) = L_i(y) + U_{i+1}(\beta y), \text{ and}$$

$$J_i^w(y) = \mathscr{L}_i(y) + \int_0^\infty (1-e_i+e_i t)\ W_{i+1}\left(\frac{\beta y}{1-e_i+e_i t}\right)\ dF_i(t)$$

where the $J_i(y)$ represent the expected value of costs over periods i, i+1,...,n, after ordering. Let S_i' be the value of y which minimize $c_i(y) + J_i^u(y)$. The minimum of this convex function is known to exist so that we can apply (10), and

$$S_i' = \sup\left\{y : \frac{dJ_i^u(y)}{dy} \leq -c_i\right\}.$$

The optimal policy in period i for the Standard Model is the S_i'-policy

order $S_i' - x_i$ if $x_i < S_i'$

order nothing if $x_i \geq S_i'$.

Lemma 2 and Corollary 1 to follow are at most small deviations from known results in inventory theory.

<u>Lemma 2</u>. $U_i'(x) = \max\{-c_i, \frac{d}{dx} J_i^u(x)\}$.

$W_i'(x) = \max\{-c_i, \frac{d}{dx} J_i^w(x)\}$.

Proof. We will give the proof for the first equation only, as both equations are proved in the same way. By the optimality of an S_i'-policy just mentioned,

$$U_i'(x) = -c_i \qquad \text{if} \qquad x_i \leq S_i'$$

$$= \frac{d}{dx} J_i^u(x) \qquad \text{if} \qquad x_i > S_i'.$$

Thus all that is needed to conclude the proof is that $\frac{d}{dx} J_i^u(x) \leq -c_i$ for $x \leq S_i'$ and $\frac{d}{dx} J_i^u(x) \geq -c_i$ for $x > S_i'$. This follows from the definition of S_i' and the fact that $\frac{d}{dx} J_i^u(x)$ in increasing. Q.E.D.

Corollary 1. $S_i' = \sup\{x:U_i'(x) \leq -c_i\}$ and $S_i = \sup\{x:W_i'(x) \leq -c_i\}$ where S_i is the critical level described after Theorem 1.

At this point we will assume that $\Delta_i > 0$ with probability one for $i=1,\ldots, n+1$. Two sufficient conditions for this to hold are that all the $e_i < 1$, or that all the $A_i > 0$ with probability one.

Lemma 3. For $i=1,\ldots,n$, and all x,

$$E_{\Delta_i}[W_i'(x/\Delta_i)] \geq U_i'(x).$$

Proof. The result holds for $i=n+1$ by (5) and the assumption that $\Delta_{n+1} > 0$. We assume that it holds for $i+1$ and shows that it holds for i. By Lemma 2, $E_{\Delta_i}[W'_i(x/\Delta_i)] = E_{\Delta_i} \max\left\{-c_i, \frac{d}{dx} J_i^w(x/\Delta_i)\right\}$. Since $T(x) \triangleq \max\{-c_i,x\}$ is a convex function, Jensen's inequality implies that $E_{\Delta_i} \max\left\{-c_i, \frac{d}{dx} J_i^w(x/\Delta_i)\right\} \geq \max\left\{-c_i, E_{\Delta_i}\left(\frac{d}{dx} J_i^w(x/\Delta_i)\right)\right\} = \max\left\{-c_i, E_{\Delta_i}\left[\mathscr{L}_i'(x/\Delta_i) + \int_0^\infty \beta W'_{i+1}\left(\frac{\beta x}{\Delta_i(1-e_i+e_it)}\right) dF_i(t)\right]\right\}$. By Lemma 1 and (8), $E_{\Delta_i}(\mathscr{L}_i'(x/\Delta_i)) = L_i'(x)$, and by the induction hypothesis $E_{\Delta_{i+1}}\left(W_{i+1}'\left(\frac{\beta x}{\Delta_{i+1}}\right)\right) \geq U_{i+1}'(\beta x)$. Therefore $E_{\Delta_i}(W_i'(x/\Delta_i))$ is greater than or equal to $\left\{-c_i, L_i'(x) + \beta U_{i+1}'(\beta x)\right\} = \max\left\{-c_i, \frac{d}{dx} J_i^u(x)\right\} = U_i'(x)$.

Q.E.D.

Lemma 4. Let f and g be two nondecreasing functions on R into R and K be a positive constant. Let $x_f = \sup\{x : f(x) \leq c\}$, and $x_g = \sup\{x : g(x) \leq c\}$ be finite. If $g(x/K) \geq f(x)$, for all x, then $x_f \geq x_g K$.

Proof. Assume the contrary that $x_g K > x_f$, so that there is an $\varepsilon > 0$ such that $x_g K > x_f + \varepsilon$. We have $f(x_f+\varepsilon) > c$ and $c \geq g(x_g) \geq g\left(\frac{x_f+\varepsilon}{K}\right)$. Those two inequalities contradict $g\left(\frac{x_f+\varepsilon}{K}\right) \geq f(x_f+\varepsilon)$. Q.E.D.

Theorem 2. Recall that $\alpha=0$. The critical level for the Dependent Demand Model in period 1, $S_1\Delta_1 = S_1\mu$, is less than or equal to the critical level for the Standard Model in period 1, S_1'. Therefore, the Dependent Demand Model results in a decision at least as flexible as the Standard Model.

Proof. By Corollary 1 $S_1^1 = \sup\{x : U_1'(x) \leq -c_1\}$ and $S_1 = \sup\{x : W_1'(x) \leq -c_1\}$. By Lemma 3 $W_1'(x/\mu) \geq U_1'(x)$. Now apply Lemma 4 to include that $S_1\mu \leq S_1'$. Q.E.D.

Theorem 2 applies to period 1. Once we get to period 2 the theorem can be reapplied to a reformulation of the Standard Model. In this reformation period 2 is interpreted as period 1, period 3 becomes period 2, etc.

Theorem 2 was established under the assumption that $\alpha=0$. The concept of flexibility suggests that Theorem 2 is also true for $\alpha>0$. However, an example in Azoury [1979] for a Bayesian inventory model suggests that the result does not hold for $\alpha>0$. Thus a conjecture for the $\alpha>0$ case appears ill-advised.

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8